Lesson 10. Poisson Processes – Decomposition and Superposition

0 Warm up

Example 1. Let's start by revisiting the Beehunter example from Lesson 9, slightly modified.

Citizens of Beehunter have complained that a busy intersection has recently become more dangerous, and they are demanding that the city council take action to make the intersection safer. After studying the intersection, a traffic engineer has determined that accidents at this intersection can be modeled as a Poisson process with a rate of 1.5 accidents per week.

a. What is the probability that the first accident occurs some time after 3 weeks but before 5 weeks?
b. What is the probability that exactly one accident occurs in the interval from 3 to 5 weeks?

1 Overview

- Decomposing a Poisson process into two arrival counting subprocesses
- Superposing (combining) two Poisson processes into one arrival counting process

2 Decomposition of Poisson processes

- Suppose arrivals occur according to a Poisson process with arrival rate λ
- Suppose that a fraction $(1-\gamma)$ of these are Type 0 arrivals, and γ are Type 1
  - e.g., major vs. minor accidents
- We can model each arrival type as a **Bernoulli random variable** with success probability γ:

$$B = \begin{cases} 0 & \text{with probability } 1-\gamma \\ 1 & \text{with probability } \gamma \end{cases}$$

- Let's assume:
  - types for all arrivals are independent and time stationary
  - types and interarrival times are independent
- The **decomposition property**:
  - Type 0 arrivals form a Poisson process with arrival rate $\lambda_0 = (1-\gamma)\lambda$
  - Type 1 arrivals form a Poisson process with arrival rate $\lambda_1 = \gamma\lambda$
  - These two processes are independent
- This works because the Poisson process is decomposed by a independent Bernoulli variables
- Other methods of decomposition do not necessarily lead to Poisson subprocesses
- For more details, see SMAS page 111

Example 2. In Example 1, suppose 80% of the accidents are minor, and 20% are major.

a. What is the probability that fewer than 4 minor accidents occur during any 4-week period?
b. What is the expected number of major accidents in any 8-week period?

Example 3. You have been asked to conduct a study of the pedestrian crossing near Chick & Ruth's in Downtown Annapolis. Assume the following behavior. Pedestrians approach the crossing at a rate of 6 pedestrians per minute; one-third of them are on the left side, and two-thirds of them are on the right side. Pedestrians wait until the walk signal, at which time all waiting pedestrians cross instantaneously. Suppose the walk signal occurs every 2 minutes.

a. What is the expected number of pedestrians crossing left to right on a given walk signal?
b. What is the probability that at least one pedestrian crosses right to left on any particular signal?

3 Superposition of Poisson processes

- We can also combine Poisson processes
- Suppose that:
  - Type 0 arrivals follow a Poisson process with arrival rate λ_0
  - Type 1 arrivals follow a Poisson process with arrival rate λ_1
  - These processes are independent of each other
- The **superposition property**: the arrivals from both processes together form a Poisson process with arrival rate $\lambda = \lambda_0 + \lambda_1$
- This works because the two Poisson processes are independent
- For more details, see SMAS pp. 111-112

Example 4. The Bank of Simplexville opens at 8 a.m. Customers arrive at the lobby at a rate of 10 per hour. The bank also has a separate ATM where customers arrive at a rate of 20 per hour. Approximate these arrival processes as Poisson processes. What is the probability that the 100th bank customer (at the lobby or the ATM) arrives before noon, given that 50 customers have arrived by 10 a.m.?

Example 5. The Markov Company has two salespeople, John and Louise. On average, John receives 8 orders per week, while Louise receives 12 per week. Suppose the orders arrive according to a Poisson process.

a. What is the probability that the total sales for two weeks will be more than 30 orders?
b. What is the expected number of orders in one month?

4 Exercises

Problem 1 (SMAS Exercise 5.3)**.** Patients arrive at a hospital emergency room at a rate of 2 per hour. A doctor works a 12-hour shift from 6 a.m. until 6 p.m. Answer the following questions by approximating the arrival-counting process as a Poisson process.

a. Of patients admitted to the emergency room, 14% are classified as "urgent." What is the probability that the doctor will see more than six urgent patients during her shift?

b. The hospital also has a walk-in clinic to handle minor problems. Patients arrive at this clinic at a rate of 4 per hour. What is the probability that the total number of patients arriving at both the emergency room and clinic from 6 a.m. until 12 noon will be greater than 30?

Problem 2 (SMAS Exercise 5.10)**.** When a power surge occurs on local electric lines, it can damage a computer plugged into the line if the computer does not have a "surge protector." There are various types of surges: "Tiny" surges occur at the rate of 8 per hour, but they cannot damage a computer. "Small" surges occur at the rate of 1 every 18 hours; a small surge will damage an unprotected computer with probability 0.005. "Moderate" surges occur at a rate of 1 every 46 hours; a moderate surge will damage an unprotected computer with probability 0.08. Suppose that the arrival of each type of surge can be modeled as an independent Poison process.

a. What is the expected number of surges of any type during 8 hours of computer work?
b. What is the expected number of computer-damaging surges during 8 hours of computer work?
c. What is the probability that there will be no computer-damaging surge during 8 hours of computer work?

Problem 3 (SMAS Exercise 5.12, modified)**.** Intellibabble is a fledgling generative AI company that provides two products to its customers: a chatbot and an image generator. Requests for the chatbot arrive a rate of 400 per hour: requests for the image generator arrive at a rate of 1000 per hour. Suppose that both arrival processes are well modeled as independent Poisson processes.

a. All incoming requests are initially handled by a single front-end server. What is the probability that this server receives more than 2000 requests between 1 p.m. and 2:30 p.m.?

b. Suppose that the front-end server distributes the requests randomly between two processing servers (called A and B) that fulfill the requests in such a way that each of the processing servers is equally likely to receive a new request. Assuming that the time for the front-end server to do this is essentially 0, what is the probability that both of the processing servers receive more than 1000 requests between 1 p.m. and 2:30 p.m.?

Problem 4. Automobiles pass a point on the highway at a rate of 1 per minute. If 5% of all automobiles are trucks and the arrival process is well approximated by a Poisson process, then answer the following questions:

a. What is the probability that at least one truck passes by during an hour?

b. Given that 10 trucks have passed by in an hour, what is the expected total number of automobiles that have passed by in that time?

Problem 5 (SMAS Exercise 5.15)**.** This problem concerns capacity planning for a manufacturing company. The company has two salespersons, John and Louise, who each cover one half of the United States. At the end of each week, the salespersons report their sales to the company, which then manufactures the products that have been ordered.

The company has three products, which it calls A, B and C for simplicity. Each salesperson obtains 10 orders per week, on average, of which approximately 20% are for A, 70% are for B, and 10% are for C. In terms of capacity, it takes 25 person-hours to produce one A, 15 person-hours to produce one B, and 40 person-hours to produce one C.

Help the company do its capacity planning by answering the following questions. You may assume that the arrival of orders to each salesperson can be well approximated as a Poisson process.

a. A Poisson process follows the stationary-increments property. What is the physical interpretation of the stationary-increments property in this situation?
b. What is the probability that the total sales for 1 week will be more than 30 products?
c. Capacity can only be changed on a monthly basis. What is the expected number of person-hours the company will need over a 1-month period?
d. What is the probability that Louise will sell more than 5 product Bs on each of 2 consecutive weeks?